

# Hawking Radiation in Trace Anomaly Free Frames

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## Abstract

We have used the results of renormalization of a two-dimensional quantum stress tensor to develop a conformally invariant dynamical model. The model requires the consideration of those conformal frames in which there exists a correspondence between the trace anomaly and a cosmological constant. We apply this model to a two dimensional-Schwarzschild (-de Sitter) spacetime to show that in these conformal frames one may achieve Hawking radiation without recourse to the trace anomaly .

## 1 Introduction

In the semiclassical approximation of quantum gravity [1] the expectation value of a quantum stress-tensor is used as the source of the Einstein field equations. In order to have a well defined stress-tensor, renormalization must be carried out due to intrinsically divergent nature of products of field operators at the same spacetime point. The remarkable consequences of all the renormalization programs which have been developed so far is that the stress-tensor of a conformally invariant field obtains a nonvanishing trace [2]. There have been many attempts to understand the physical implications of such an anomalous trace. Specifically, it has been shown [3] that the imposition of a regularity condition of a covariantly conserved stress tensor on the horizon of a Schwarzschild black hole would lead to a connection between the trace anomaly and Hawking radiation in two dimensions. In view of this observation one may attribute a nonlocal characteristic to the trace anomaly in the sense that it relates the properties of the quantum stress tensor on the horizon to those all those at null infinity. The nonlocal characteristic of the trace anomaly has been already investigated in some different context [5, 6]. Here we wish to study this property by introducing a two-dimensional conformally invariant dynamical model. The use of this model in the two-dimensional Schwarzschild (-de Sitter) spacetime reveals that the existence of the trace anomaly may not be a necessary condition for obtaining the Hawking effect.

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We have organized this paper as follows: In section 2, we use the results of renormalization of a quantum stress-tensor in two-dimensions to develop a conformally invariant dynamical model. The conformal invariance allows us to consider the model in different conformal frames. In particular, we shall obtain a consistency requirement enforcing the consideration of those frames in which the trace anomaly takes the constant value  $\Lambda$ . In section 3, we first apply this model to a two dimensional Schwarzschild spacetime by considering a class of conformal frames characterized by  $\Lambda = 0$ . By exploring the conservation law of the quantum stress-tensor in these frames, in which the trace anomaly vanishes, we obtain an outward flux of thermal radiation with temperature  $kT_H = (8\pi M)^{-1}$ . We then bring the case  $\Lambda \neq 0$ . We shall show that in the corresponding conformal frames the Hawking temperature receives a correction term as a result of the existence of  $\Lambda$ . In section 4, we offer some concluding remarks. We shall work in units in which  $c = \hbar = 1$ , and our sign conventions are those of Hawking and Ellis [7].

## 2 The model

We consider a massless quantum scalar field conformally coupled to a two-dimensional gravitational background. Renormalization of the corresponding stress-tensor  $\Sigma_{\mu\nu}$  leads to the following results [8]

$$\nabla^\mu \Sigma_{\mu\nu} = 0 \quad (1)$$

$$\Sigma^\mu_\mu = \frac{1}{24\pi} R \quad (2)$$

where  $\nabla$  denotes the covariant differentiation operator, and  $R$  is the curvature scalar. The first equation indicates that  $\Sigma_{\mu\nu}$  is covariantly conserved. The second one is the trace anomaly coming from the renormalization process, and suggest that  $\Sigma_{\mu\nu}$  can be written in the form

$$\Sigma_{\mu\nu} = \Sigma_{\mu\nu}^{(0)} + \frac{1}{48\pi} g_{\mu\nu} R \quad (3)$$

where  $\Sigma_{\mu\nu}^{(0)}$  is a traceless tensor. The plane is to construct a dynamical model based on the decomposition (3), in the close correspondence to [5, 6]. The essential step is to identify  $\Sigma_{\mu\nu}^{(0)}$  with the stress-tensor of a conformally invariant scalar field

$$T_{\mu\nu}[\phi] = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\gamma \phi \nabla^\gamma \phi \quad (4)$$

where  $\phi$  is a C-number scalar field, satisfying

$$\square \phi = 0. \quad (5)$$

The traceless condition of  $T_{\mu\nu}$  is satisfied. The relation (3) then leads

$$\Sigma_{\mu\nu} = T_{\mu\nu} + \frac{1}{48\pi} g_{\mu\nu} R \quad (6)$$

This may now be interpreted as a general condition imposed on  $\Sigma_{\mu\nu}$  for a fixed background geometry. Nevertheless, from comparing Eq.(1) with Eq.(6) one infers that such an interpretation is not consistent on the background metric, for the tensor  $T_{\mu\nu}$  can't be conserved and receives a source term given by the trace anomaly. This discrepancy may be removed by appealing to the conformal invariance of the above model, implying that the coupling of  $\phi$  to geometry can be studied in many different conformal frames which are dynamically equivalent. A generic question which appears in these situation is that which of these frames corresponds to the physical one. In the present case we wish to find a particular conformal frame in which  $T_{\mu\nu}$  can be expressed as a conserved stress-tensor. To this end we first consider a conformal transformation

$$\bar{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} \quad (7)$$

$$\bar{\phi}(x) \longrightarrow \Omega^{-1}(x)\phi(x) \quad (8)$$

under which (7) takes the form

$$\bar{\Sigma}_{\mu\nu} = \bar{T}_{\mu\nu} + \frac{1}{48\pi}\bar{R}\bar{g}_{\mu\nu}. \quad (9)$$

We then restrict our attention to those frames in which the trace anomaly can be related to some constant value. These frames are defined by

$$\bar{R} = \Lambda \quad (10)$$

or equivalently,

$$\Omega^{-2}\{R + 2\nabla_\gamma \ln \Omega \nabla^\gamma \ln \Omega - 2\frac{\square\Omega}{\Omega}\} = \Lambda \quad (11)$$

where we take  $\Lambda$  to be a cosmological constant. This relation is actually a constraint on the conformal factor. It determines the class of conformal frames in which  $T_{\mu\nu}$  appears as a conserved tensor. This can be seen by combining the equations(9,10) which gives

$$\bar{\Sigma}_{\mu\nu} = \bar{T}_{\mu\nu} + \frac{1}{48\pi}\Lambda\bar{g}_{\mu\nu}. \quad (12)$$

Our next step is to study the relation (12) in a particular case corresponding to a two dimensional Schwarzschild(de-Sitter) spacetime.

### 3 Hawking Effect

Consider first the case  $\Lambda = 0$ , we intend to apply the above model to a two-dimensional Schwarzschild spacetime, described by the metric

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 \quad (13)$$

where  $M$  is the mass of the black hole and  $G$  is the gravitational constant. This metric can be written in the conformally flat form

$$ds^2 = h(r)(-dt^2 + dr^{*2}) \quad (14)$$

where :

$$h(r) = \left(1 - \frac{2GM}{r}\right), \quad \frac{dr}{dr^*} = h(r) \quad (15)$$

A conformal transformation transforms (14) to

$$\bar{ds}^2 = \bar{\Omega}^2(r)(-dt^2 + dr^{*2}), \quad \bar{\Omega}^2(r) = h(r)\Omega^2(r) \quad (16)$$

The corresponding conformal factor can be determined by substituting  $R = \frac{4GM}{r^3}$  and  $\Lambda = 0$  into the constraint(11) which leads to

$$\Omega^{-2}(r)\{\nabla_\gamma \ln \Omega(r)\nabla^\gamma \ln \Omega(r) + \frac{2GM}{r^3} - \frac{\square \Omega(r)}{\Omega(r)}\} = 0 \quad (17)$$

This is now a differential equation for determining the function  $\Omega(r)$ . We subject the solutions of (17) to the condition that the geometry of the quasi-flat regions ( $r_{qf} = r \gg 2GM$ ) coincides with that of the background frame, namely  $\Omega(r) \rightarrow 1$  when  $r \rightarrow r_{qf}$ . On these regions we must have  $\bar{\Omega}(r) \rightarrow 1$ . We may obtain the general form of  $\Sigma_{\mu\nu}$  by using the results of the appendix. This gives

$$\bar{\Sigma}_\mu^\nu = \bar{\Sigma}_\mu^{(r)\nu} + \bar{\Sigma}_\mu^{(eq)\nu}. \quad (18)$$

The tensors  $\bar{\Sigma}_\mu^{(r)\nu}$  and  $\Sigma_\mu^{(eq)\nu}$  are given by

$$\bar{\Sigma}_\mu^{(r)\nu} = K\bar{\Omega}^{-2}(r) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \Sigma_\mu^{(eq)\nu} = Q\bar{\Omega}^{-2}(r) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (19)$$

where  $K$  and  $Q$  are constants. the tensor  $\Sigma_\mu^{(eq)\nu}(r \rightarrow r_{qf})$  is proportional to the stress-tensor of a gas under equilibrium condition

$$\frac{\pi}{6}(kT)^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (20)$$

It indicates a thermal bath arising from the existence of a cosmological event horizon [9, 10]. In the present case, since  $\Lambda = 0$  we should have  $Q = 0$ , requiring a vanishing temperature. On the other hand, the tensor  $\bar{\Sigma}_\mu^{(r)\nu}(r \rightarrow r_{qf})$  contains off diagonal (flux) components and should be compared with the stress-tensor of thermal radiation

$$\frac{\pi}{12}(kT)^2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \quad (21)$$

It gives a temperature  $kT_H = (8\pi M)^{-1}$  if  $K = (768M^2)^{-1}$ . It means that in the conformal frame which is characterized by anomaly cancellation, one obtains an outward flux of radiation with the temperature consist with the Hawking radiation.

We now consider the case  $\Lambda \neq 0$ . In this case the back ground metric should be considered to be the Schwarzschild-de Sitter spacetime. Thus the conformal factor of the metric (13) should be taken to be

$$h(r) = \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right) \quad (22)$$

A conformal transformation transforms (14) to the form (16). Using the results of the Appendix, the general form of  $\bar{\Sigma}_{\mu\nu}$  in this case may be written as

$$\Sigma_{\mu}^{(r)\nu} = K\bar{\Omega}^{-2}(r) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + \frac{\Lambda}{48\pi} \begin{pmatrix} 1 + \bar{\Omega}^2(L)\bar{\Omega}^{-2}(r) & 0 \\ 0 & 1 - \bar{\Omega}^2(L)\bar{\Omega}^{-2}(r) \end{pmatrix} \quad (23)$$

$$\bar{\Sigma}_{\mu}^{(eq)\nu} = Q\bar{\Omega}^{-2}(r) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (24)$$

where we have used

$$H(r) = \frac{\Lambda}{48\pi} (\bar{\Omega}^2(r) - \bar{\Omega}^2(L)). \quad (25)$$

We define the quasi-flat regions as

$$2GM \ll r_{qf} \ll \frac{1}{\sqrt{\Lambda}}. \quad (26)$$

For a sufficiently small  $\Lambda$ , one may define a domain of many orders of magnitude on which (26) is valid. We may impose  $\bar{\Omega}(r) \rightarrow 1$  when  $r \rightarrow r_{qf}$  as the boundary condition. The tensor  $\bar{\Sigma}_{\mu}^{(eq)\nu}(r \rightarrow r_{qf})$  describes a thermal bath coming from the cosmological event horizon. Thus comparing it with (20) gives a temperature  $kT_c = \sqrt{\Lambda}$  if we choose  $Q = \frac{\pi}{6}\Lambda$ . The tensor  $\bar{\Sigma}_{\mu}^{(r)\nu}$  yields

$$\Sigma_{\mu}^{(r)\nu}(r \rightarrow r_{qf}) = \frac{\pi}{12}(8\pi M)^{-2} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + O(\Lambda) \quad (27)$$

if  $K = (768\pi M^2)^{-1}$ , and describes an outward radiation with temperature

$$kT_H = (8\pi M)^{-1} + O(\Lambda) \quad (28)$$

The second term on the right hand side is a correction term to the Hawking temperature, arising from the cosmological event horizon.

## 4 CONCLUDING REMARKS

A conformally invariant dynamical model is developed, using the standard results of renormalization of a quantum stress tensor in two dimensions. We found it necessary to consider the model in conformal frames in which either the trace anomaly vanishes or takes a constant value  $\Lambda$ , which is regarded to be a cosmological constant. There is a correspondence between the conformal frames and quasi-flat regions of a Schwarzschild and a Schwarzschild-de Sitter spacetime respectively. We have shown that in the conformal frames in which an anomaly cancellation takes place we obtain an outward flux of thermal radiation with the Hawking temperature. This observation emphasizes the dispensable role of the trace anomaly for obtaining the Hawking radiation in this frame. On the other hand, in a conformal frame in which the trace anomaly takes a constant value we have received a correction term to the Hawking temperature due to the presence of a cosmological event horizon.

## 5 ACKNOWLEDGMENTS

One of us (S.M) wishes to thank Y.Bisabr for helpful discussions

## 6 APPENDIX

Here we are looking for the general form of  $\bar{\Sigma}_{\mu\nu}$  which satisfies the conservation equation

$$\bar{\nabla}^\mu \bar{\Sigma}_{\mu\nu} = 0 \quad (29)$$

for the metric (15). We first note that the nonzero Christoffel symbols of (15) are

$$\Gamma_{tt}^{r*} = \Gamma_{tr*}^t = \Gamma_{r*t}^t = \Gamma_{r*r*}^{r*} = \frac{1}{2} \frac{d}{dr} \bar{\Omega}(r) \quad (30)$$

We then use the spherical symmetry condition and the fact that  $\bar{\Sigma}_{\mu\nu}$  is time independent outside the horizon to write the conservation equation (29) in the form

$$\partial_{r*} \bar{\Sigma}_t^{r*} + \Gamma_{tr*}^t \bar{\Sigma}_t^{r*} - \Gamma_{tt}^{r*} \bar{\Sigma}_{r*}^t = 0 \quad (31)$$

$$\partial_{r*} \bar{\Sigma}_{r*}^{r*} + \Gamma_{tr*}^t \bar{\Sigma}_{r*}^{r*} - \Gamma_{tr*}^t \bar{\Sigma}_t^t = 0 \quad (32)$$

where  $\bar{\Sigma}_{r*}^t = -\bar{\Sigma}_t^{r*}$ , and  $\bar{\Sigma}_t^t = \bar{\Sigma}_\gamma^\gamma - \bar{\Sigma}_{r*}^{r*}$ ;  $\bar{\Sigma}_\gamma^\gamma$  is the trace anomaly in two-dimensions. Using these equations one can write

$$\frac{d}{dr} [\bar{\Omega}(r) \bar{\Sigma}_t^{r*}] = 0 \quad (33)$$

and

$$\frac{d}{dr} [\bar{\Omega}(r) \bar{\Sigma}_{r*}^{r*}] = \frac{1}{2} \left[ \frac{d}{dr} \bar{\Omega}(r) \right] \bar{\Sigma}_\gamma^\gamma \quad (34)$$

the relation (33) gives immediately

$$\bar{\Sigma}_t^{r*} = \alpha \bar{\Omega}^{-1}(r) \quad (35)$$

with  $\alpha$  being an integration constant. Equation(34) gives

$$\bar{\Sigma}_{r*}^{r*}(r) = (H(r) + \beta) \bar{\Omega}^{-1}(r), \quad \beta = \bar{\Omega}(L) \bar{\Sigma}_{r*}^{r*}(L) \quad (36)$$

where  $L = 2GM$ , and

$$H(r) = \frac{1}{2} \int_L^r \bar{\Sigma}_\gamma^\gamma(r') \frac{d}{dr'} \bar{\Omega}(r') dr' \quad (37)$$

From the equation(35)and (36), we may now write the general form of  $\bar{\Sigma}_\mu^\nu$  as

$$\bar{\Sigma}_\mu^{(r)\nu} = \begin{pmatrix} \bar{\Sigma}_\gamma^\gamma - \bar{\Omega}^{-1}(r)H(r) & 0 \\ 0 & \bar{\Omega}^{-1}(r)H(r) \end{pmatrix} + \bar{\Omega}^{-1}(r) = \begin{pmatrix} -\beta & -\alpha \\ \alpha & \beta \end{pmatrix} \quad (38)$$

If we define  $Q = \beta - \alpha$  and  $K = \alpha$ ,  $\bar{\Sigma}_\mu^\nu$  takes then the following form

$$\bar{\Sigma}_\mu^\nu = \bar{\Sigma}_\mu^{(r)\nu} + \bar{\Sigma}_\mu^{(eq)\nu} \quad (39)$$

with

$$\bar{\Sigma}_\mu^{(r)\nu} = \begin{pmatrix} \bar{\Sigma}_\gamma^\gamma - \bar{\Omega}^{-1}(r)H(r) & 0 \\ 0 & \bar{\Omega}^{-1}(r)H(r) \end{pmatrix} + K\bar{\Omega}^{-1}(r) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \quad (40)$$

$$\bar{\Sigma}_\mu^{(eq)\nu} = Q\bar{\Omega}^{-1}(r) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (41)$$

The tensor  $\bar{\Sigma}_\mu^{(eq)\nu}$  is the traceless part of  $\bar{\Sigma}_\mu^\nu$  and only  $\bar{\Sigma}_\mu^{(r)\nu}$  contains off diagonal components.

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